In Search of Lost Spaces: A spatial fine-grained way

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Without semantics, logic would be a meaningless stream of symbols and tasteless rules over them. While a syntactical perspective simply sees logic as practices that we do follow a certain class of rules, a semantical approach seems to say something more contentful: *what (mathematical) structure makes such reasoning valid*.

Between syntax and semantics, two directions of interplay are observed. For one direction, logic (syntax) employs semantics as handy *models*, which are more understandable than a bunch of rules. We can accommodate a familiar and well-known mathematical structure to a given logic and study the logic within the terminology and framework of that mathematical structure, borrowing previous results of some field of pure mathematics. For example, first order predicate logic enjoys set theory as its semantic system as its common semantic terminology. The other direction is that *structure used as a semantics uses syntax*. We can study a certain structure *by logically analyzing* it. The latter direction is crucial for a certain class of philosophers, especially who do *metaphysics* (cf. [5]); Due to its *empirically independent* nature –you cannot do experiments of metaphysics as a physicist do for physics– logic seems to be the last guideline remaining at metaphysicians' hands to indirectly "observe" metaphysical reality.

Our ultimate and metaphysical target of this article is *space*. When we are curious about space so broadly conceived that it does *not* have to be physical space which is accessible empirically, what can logicians do? Modal logic could do some. To this end, instead of our most common and popular semantics –*relational* or *Kripke* semantics, we employ its historically antecedent alternative — *topological semantics* (cf. [2]). Recent revival of this semantics with spatial flavor is (partly but largely) motivated by this need of *logics of space* —logical correspondences of several concepts of space (cf. [6], [1]) .

Nevertheless, the current framework of topological semantics does *not* satisfy this philosophical request. Recall that the standard topological interpretation sees necessity as *interior* and possibility as *closure*. This is *too coarse*; namely, it fails to distinguish, characterize, or *define* common topological/spatial characteristics. In fact, the classical result of McKinsey and Tarski says that modal logic $S4$ is the logic of not only real line R but also any metric, separable and dense-in-itself space, and *even further, any topological space*. This result may be formally beautiful but is disappointing for our current purpose: to understand space via logic. For instance, a topological feature *connectedness* cannot be characterized under the language of propositional modal logic with the standard topological interpretation. You cannot use any logic weaker than $S4$ either to understand more about space. Even worse: basic modal logics cannot define separation theorems $T_i (i \leq 6)$ (cf. [3])!

To distinguish and define more kinds of space via logic, logicians have taken two approaches with modest modifications. The first one is to expand the vocabulary (e.g. adding extra operators such as universal modality in addition to local one). The second one is to change interpretation (e.g. to see possibility as derivative instead of closure).

This presentation takes the third way: to provide a brand-new semantics for our spatial endeavor. The new semantics is called *spatial* semantics (for the time being), which features spatial notions such as *regions* and *dimensions*.

These two major modifications expand our logical descriptions of space. The formal concept of

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a possible world w used to be a point of topological space $w \in X$ with a topological structure $\langle X, \tau \rangle$. This unquestioned setting has made classical logicians' lives easier; it is enough to settle $||\neg p||$ as the compliment of $||\neg p||$ to behave classically well. Such a swallow of classical logic has hurt non-classical persons. But no more under my regime; Since our possible world as a region can cross the borderline between $||p||$ and $||\neg p||$, we need some topological or spatial constraints over models to control behavior, making the law of excluded middle $p \vee \neg p$ valid again.

The other change –introducing *dimension*– affects on how to interpret modality. Our spatial model is built on a *product* of topological spaces in the form of $M^i = \langle \prod_{i \in I} \langle X_i, \tau_i \rangle, V \rangle$. In order to check whether a world $w_i \in W$ does satisfy p in a model M^i, we need to consult the model(s) M_j^{i-1} "one-dimension down", generated –in a more intuitive term– *squeezed* by *projection* function $\pi_{j \in I} : \prod_{i \in I} M^i \mapsto M_j^{i-1}$. This mocks our expectation over necessity: a proposition p is *safely* true as similarly as topological semantics does with interior (i.e. not on the edge).

Upon introducing the new semantics, this article displays several partial results toward determining logics by spatial conditions over models. Examples include something which makes a influential philosopher David Lewis smile: $||p||$ and $||\neg p||$ form a *separation* for each proposition p and each possible world is topologically *connected* (with itself), as *concrete modal realism* insists in [4]. Moreover, the number of *depth* of modal operators tells *how many dimensions* its corresponding space should have.

The closing section will showcases remaining issues. The most crucial one is of *heuristic*; To gain results given, I have been hand-picking conditions upon countless try-and-error attempts. Following the manner of the previous topological semantics, it seems working if we find a counterpart of (topological) *bisimulation*, say, *dim-bisimulation* with a hope to bind my semantics to other previous semantics including relational semantics. A sketch that my semantics has desired properties (such as finite model property) once we have such a bisimulation-like correspondence will be given.

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