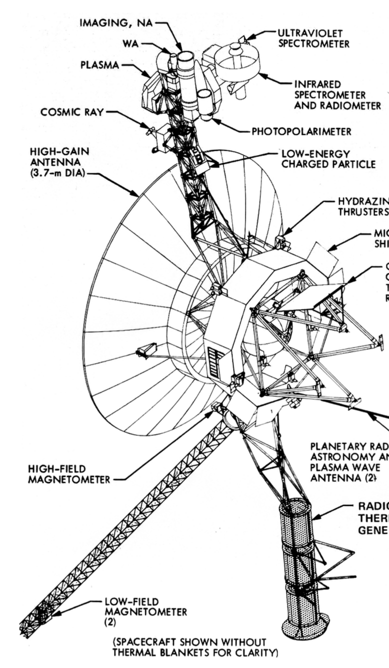
What (modal) logics say to metaphysics

A logical endeavor toward a dimensional space of multi-verses.



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Introduction [1]

Without semantics, logic [2] becomes just a stream of meaningless symbols derived via tasteless rules. A formal semantics assigns what such expressions mean – or what mathematical structure they correspond to. However, since a semantics is a mere mathematical structure, the quest keeps going: *how* or *what makes* such a mathematical structure give(s) a meaning to a sentence (or any syntactically accepted expression)?

Metaphysicians have intended [3] to provide a *metaphysical* account or description of formal semantics. To have a nice metaphysical theory, we check its formalized structure–formal semantics for well-known logics are to be examined.

1 Semantics available in the current modal market

Why do we need another formal semantics? We already have several options [2]. Each enjoys its own good points (see the table below). Nevertheless, none of them meets our needs. To begin with, relational semantics of Kripke leaves a metaphysical mystery: what is the very thing called relation in Kripke structure, metaphysically speaking? Topological semantics makes more metaphysical sense for the structure of inter-worlds space, but *too coarse* for its $S4$ completeness, indicating its incapability to distinguish logics weaker than $S4$.

Semantics	Advantages	Disadvantages
Relational [3, 1]	User-friendly	Metaphysically mysterious, classical and standard
Algebraic [?]	Importing algebraic technique	Syntax in disguise.
Topological [5]	Metaphysically making sense	Too coarse (S4-complete)
Neighborhood [4]	Fine-grained	Still mysterious

Our new semantics – named *spatial semantics* should be:

- *metaphysically making more sense* and
- *fine-grained* enough to distinguish non-classical and non-standard logics.

2 Semantics

Definition 1 (Language of PML). *Let $PROP$ be a set of propositional letters $p_0, p_1, ...$ (at most countable). A sentence ϕ of propositional modal logic (PML) is defined in a standard inductive manner:*

$$\phi ::= p_i|\neg\phi|\phi \wedge \phi|\phi \vee \phi|\phi \to \phi|\Box\phi|\Diamond\phi$$

Definition 2 (Structure of spatial semantics: locus). *Let I is an index set of at most countable. The structure of spatial semantics is called the* locus*: $L = \Pi_{i\in I}\langle D_i, \tau_i\rangle$, while each $\langle D_i, \tau_{i\in I}\rangle$ forms a topology. A world $w \in L$ forms a set of worlds $W = \{w_i|w \in L\}$.* [4]

Definition 3 (Model of spatial semantics). *A model of spatial semantics M is the form of $\langle L, V\rangle$ with L a locus defined just above and a function valuation as follows. $V : PROP \mapsto \mathcal{P}L$; with $p \in PROP, V(p) \subseteq L$.*

Key operation: squeezing

This central operation to define $\Box$ and $\Diamond$ is *squeezing*, which generates new models from a given model via its *projection*, a well-known operation on product sets (or topologies). This operation forces the model to go *one step down, in a dimensional sense* in the following manner.

Definition 4 (Projection). *Let I, J be index sets. Write X_I for $X_I = \Pi_{i\in I}X_i$. A projection on X_I with $J \subset I$ is a function $\pi_J : X_I \mapsto X_J$, $x_{i\in I} \mapsto x_{j\in J}$. Write $\overrightarrow{x} = (x_1, x_2, ..., x_i, ...)$, with $x_i \in X_i$.*

Our operation squeezing is based on a very simple type of *projection*: just eliminating one axis out of a given coordinate.

Definition 5 (Squeezing and unsqueezing). *Given $i \in I$ and $\overrightarrow{x} = (x_1, x_2, ..., x_{i-1}x_i, x_{i+1}...)$, squeezing is a function $\Downarrow_i$ which gives $\Downarrow_i \overrightarrow{x} = \overrightarrow{x} = (x_1, x_2, ..., x_{i-1}, x_{i+1}...)$ For a subset X of L, write $\Downarrow_i X = \{\Downarrow_i \overrightarrow{x}\,\overrightarrow{x} \in X\}$. Unsqueezing is defined as its* inverse. *Write $\Downarrow_i^{-1} := \Uparrow_i$.*

Let us observe examples to see how squeezing and unsqueezing work. $M, w_a \vDash p$ because $w_a \in [\![p]\!]$. Where does it make $\neg p$ true? It does *not* have to be the compliment of $[\![p]\!]$ in fact $w_b \notin [\![p]\!]$ but $w_b \nvDash \neg p$ since $w_b \notin [\![\neg p]\!]$. $M, w_c \vDash \neg p$ because $w_c \in [\![\neg p]\!]$.

To see modality, observe w_d (in a different picture but the same model M). $M, w_a \vDash \Box p$ since it has a direction to squeeze (namely $\Downarrow_1$) which makes $w_a \in \Uparrow ((\Downarrow [\![p]\!])^c)$. In contrast, $M, w_d \nvDash \Box p$ since in any direction $i \in I = \{1, 2\}$ to squeeze $\Downarrow_i w_d \notin (\Downarrow_i ([\![p]\!]^c)^c)$

There are two types of models in my framework: squeezed and original. This distinction will play a crucial role to distinguish between minimal and intuitionistic logic (under singleton conditions).

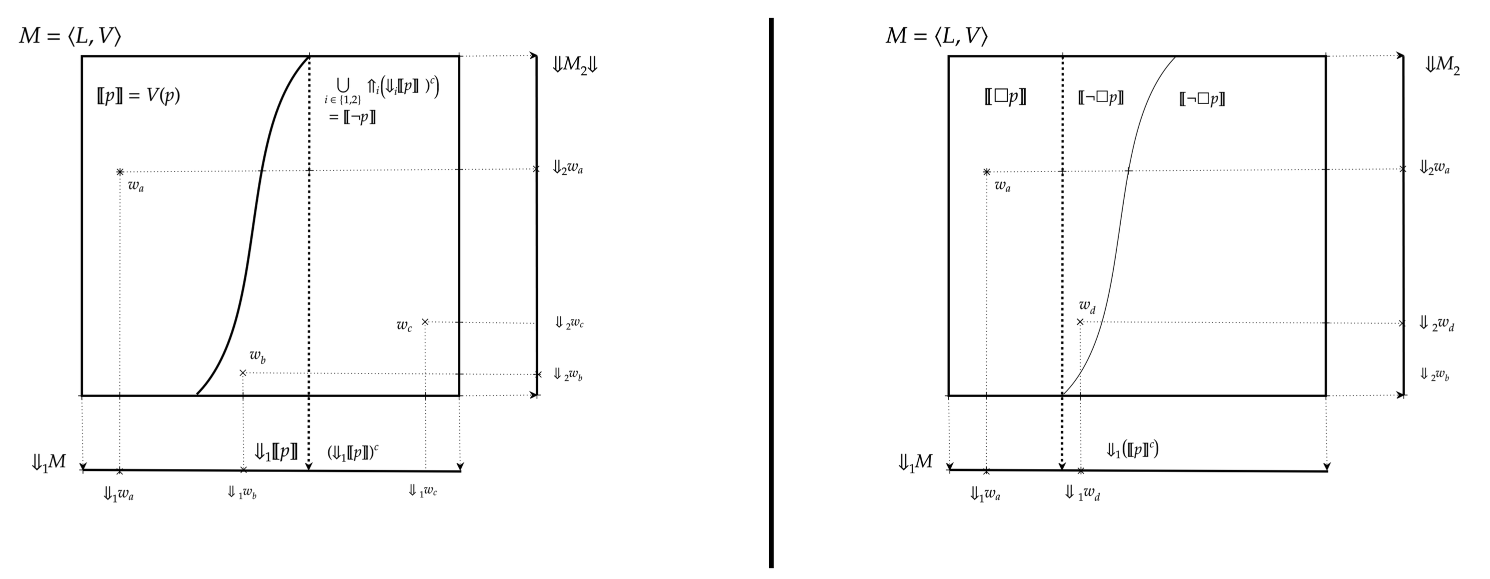
Definition 6 (Squeezed and original). *If a model is made by squeezing, it is a squeezed model. Otherwise, it is called the original model.*

Truth conditions

Definition 7 (Truth-making area). *Consider a spatial model $M = \langle L, V\rangle$. The truth-making area of a sentence ϕ is defined in the following inductive manner.*

- $[\![p]\!]_M = V(p)$
- $[\![\bot]\!]_M = [\![\phi]\!]_M \cap [\![\neg\phi]\!]_M t$
- $[\![\phi \wedge \psi]\!]_M = [\![\phi]\!]_M \cap [\![\psi]\!]_M$
- $[\![\phi \vee \psi]\!]_M = [\![\phi]\!]_M \cup [\![\psi]\!]_M$
- $[\![\phi \to \psi]\!]_M = \cup_{i\in I} \Uparrow_i^{(} (\Downarrow_i [\![\phi]\!]_M)^c \cup (\Downarrow_i ([\![\psi]\!]^c_M))^c)$
- $[\![\neg\phi]\!]_M = \cup_{i\in I} \Uparrow_i ((\Downarrow_i [\![\phi]\!])^c)$
- $[\![\Box\phi]\!]_M = \cup_{i\in I} \Uparrow_i ((\Downarrow_i ([\![\phi]\!]_M{}^c))^c)$
- $[\![\Diamond\phi]\!]_M = \cap_{i\in I} \Uparrow_i (\Downarrow_i [\![\phi]\!]_M)$

Definition 8 (Truth-condition). $M, w \vDash \phi$ *iff* $w \in [\![\phi]\!]_M$.



Metaphysical interpretation of spatial semantics

Read this semantics as Takashi Yagisawa's *dimensional modal realism*, featuring:

- modal indices as a (certain but not privileged) kind of metaphysical indices such as spatial and temporal indices
- worlds as slices of indices (and metaphysically fundamental difference between worlds W and locus L),
- impossible worlds (w s.t. $w \vDash p \wedge \neg p$) in addition to possible worlds.

3 Demonstration: make classical logic from nothing!

We can control the strength of logic by putting *spatial* constrains over our spatial models.

Claim 1 (Empty model). $\emptyset \nvDash \phi$ *for any sentence* ϕ.

Proof. Because $\emptyset \notin \emptyset$. □

Claim 2 (Failure of explosion). *Given ϕ a sentence of propositional modal logic and M^m is not empty, $M^m \nvDash \bot \to \phi$.*

Proof. For instance, consider a squeezed model $\Downarrow_2 M$ in the previous example. $\Downarrow_2 M \neq [\![\bot \to \phi]\!]$ since $\Downarrow_1\Downarrow_2 [\![\bot]\!] =\Downarrow_1\Downarrow_2 [\![]\!] =\Downarrow_1\Downarrow_2 M$. So its complement of singleton is $\emptyset$. $\Uparrow_1 \emptyset = \emptyset$. So $[\![\bot \to \phi]\!]$ is calculated in effect as $\Uparrow_1\Downarrow_1 [\![\phi]\!]$, which does *not* have to equal to the entire $\Downarrow_2 M$. □

Claim 3 (Recovery of explosion). *If we consider any non-empty model M^i which is original, $M^i \vDash \bot \to \phi$ for a sentence ϕ.*

Proof. Observe that $[\![\bot]\!] = emptyset$ in any original model M^i. So is any squeezed model (except for empty one) $\Downarrow_j M^i$, $\Downarrow_j [\![\bot]\!] = \emptyset$, implying that $(\Downarrow_j [\![\bot]\!])^c =\Downarrow_j M$. This leads that $\Uparrow_j (\Downarrow_j [\![\bot]\!])^c = M$. Therefore, no matter what $\Uparrow_j\Downarrow_j [\![\phi]\!]$ takes, $[\![\bot \to \phi]\!] = M^i$. □

Logic	Characteristic axiom	Condition
Nihil	Nothing provable	No condition at all (empty world accepted!)
Minimal		Dimensions $I \geq 0$
Intuitionistic	Explosion $\bot \to \phi$	Non-squeezed
Classical	Bivalence $P \vee \neg P$	Right-angled: There is i s.t. $\Uparrow_i\Downarrow_i [\![P]\!] = [\![P]\!]$
K	(Dual. $\Box P\neg\Diamond\neg P$)	By definition.
K	(Nec. $\vDash \phi$ implies $\vDash \Box\phi$)	Worlds are dense in locus: $L = W$.
K	(Dist. $\Box(P \wedge Q) \to (\Box P \wedge \Box Q)$)	?
T	$\Box P \to P$	Number of dimensions should be 0 or 1.
4	$\Box P \to \Box\Box P$	Number of dimensions?

Forthcoming Research

- Heuristic methods for finding spatial conditions (like Sahlqvist theorem [1] for relational structure)
- Importing *locale* (*pointless topology*) to enhance fine-grainedness and to rescue our metaphysical intuition: our world in which we live cannot be a *point*.

References

[1] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge University Press, 2002.

[2] Patrick Blackburn and Johan van Benthem. Modal logic: a semantic perspective. In Patrick Blackburn, editor, *Handbook of Modal Logic*, pages 1–84. Elsevier B.V., 2007.

[3] Saul A. Kripke. Semantical Considerations on Modal Logic. *Acta Philosophica Fennica*, 16:83–94, 1963.

[4] Eric Pacuit. *Neighborhood Semantics for Modal Logic*. Short Textbooks in Logic. Springer International Publishing, 2017.

[5] Johan van Benthem and G Bezhanishvili. Modal Logics of Space. In Marco Aiello, Ian Pratt-Hartmann, and Johan van Benthem, editors, *Handbook of Spatial Logics*, chapter 5, pages 217 – 298. Springer, Dordrecht, The Netherlands, 2007.

[1] A poster presented at the annual meeting of Japan Association for Philosophy of Science (JAPS), Chiba University, 17 June, 2018. The latest version is available on: https://www.overleaf.com/read/rxskqrbkwdqm
[2] syntactically defined as a set of axioms and inference rules and written in formal expression
[3] Discussed in the talk given in my talk given June 16, 2018.
[4] Metaphysically, $w \subseteq L$ should be better but for the sake of formal simplicity, let it be $\in$ for the time being.