Spatial semantics and characterization of non-classical logics (to come) [1]

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This talk introduces *spatial semantics*, a new semantics for propositional modal logic (PML). Furthermore, we will address the problems of characterizations of classical logic.

WHERE IS THE SPACE FOR THIS NEW SEMANTICS? For the *metaphysical* sake, this semantics is expected to work as a formal description of my metaphysical claim *spatialism*, aiming to express any entities (e.g. proposition, possible worlds) in the system in spatial terms such as location or dimension. *Formally*, this semantics is desired to have an expressive power which can capture and distinguish several systems abandoned in the previous attempts.

HOW SPATIAL? Worlds are located spreading over some *space* [3], say, *locus* [4]. Moreover, spatial semantics preserves worlds' own spatial structures in its resident locus. In addition to standard components, spatial semantics a new operation *squeezing* to construct *squeezed* models to provide truth conditions of modal operators $\Box$ and $\Diamond$ (and even $\neg$).

Definition 1 (Language of propositional modal logic (PML)) [5].
$\phi ::= P|\phi|\neg\phi|\phi \vee \phi|\phi \wedge \phi|\phi \to \phi|\Box\phi|\Diamond\phi$

Definition 2 (Structure) *A model of spatial semantics is written* $M = \langle L, W, V \rangle$ *with the followings: The* locus $L = \prod_{i \in I} D_i$ *with each* dimension D_k *a topological space* [6], *the set of worlds* [7] $W = \{w | w \subseteq U\} \subseteq \mathcal{P}L$, *and the valuation function* $V : PROP \mapsto \mathcal{P}L$.

Definition 3 (Operation: Squeezing) [8] *Consider an arbitrary spatial model* M *with* $L = \prod_{i \in I} X_i$, *and take an arbitrary subset* [9] X *of* L. *Write it in the form of* coordinate*:* $X = \{x | x = x_1 \in D_1, x_2 \in D_2, ..., x_k \in D_k, ...\}$. *Pick an arbitrary dimension* D_k. *The squeezed (to the direction of* D_k*) is generated via the following function* $\Downarrow^k$ *called* squeezing*:*
$\Downarrow^k X = \{\Downarrow^k x | \Downarrow^k x = x_1 \in D_1, x_2 \in D_2, ..., \cancel{x_k \in D_k}, ...\}$.

Definition 4 (Non-contradictory-well-behavingness) *A model* M *and a world* w *of the model* M *is called* non-contradictory-well-behaving *with respect to a sentence* ϕ *when for any subformula* ψ *of* ϕ, w *in* M *does* not *make true* both *of* ψ *and* $\neg\psi$ *at the same time.*

[1] This talk is taken from chapters of my master thesis Spatial Modal Realism (supervisors: Franz Berto and Nick Bezhanishvili). The latest version of this handout is available online at `https://www.overleaf.com/read/dmjfcffyxdkz`

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[3] This feature is an inherit of topological semantics but with some dimensional twist.

[4] This terminology is from Aristotle's *Physics*.

[5] $P \in PROP$, $PROP$ a set of propositional letters.

[6] Precisely, write $\langle D_k, \tau_k \rangle$ when specification needed

[7] Note that worlds do not have to be mere points but subsets of the universe.

[8] Squeezing is just a fancy name for operation *projection* on product topology.

[9] Note that anything in the structure, viz., world, valuation of a proposition, is *subset* of L in spatial semantics.

Definition 5 (Truth conditions: a starter kit) • *$M, w \vDash P$ iff $w \subseteq [\![P]\!]^M$,*

- *$M, w \nvDash \phi$ iff w in M does* not *satisfy the truth condition of ϕ,*
- *$M, w \vDash \phi \wedge \psi$ iff $M, w \vDash \phi$ and $M, w \vDash \psi$,*
- *$M, w \vDash \phi \vee \psi$ iff $M, w \vDash \phi$ or $M, w \vDash \psi$,*
- *$M, w \vDash \phi \to \psi$ iff $M, w \vDash \Box(\neg\phi \vee \psi)$,* [10]
- *$M, w \vDash \neg\phi$ iff $M, w \vDash \phi \to \bot$, and*
- *$M, w \vDash \bot$ iff $M, w \vDash \phi \wedge \neg\phi$.*

Definition 6 (Truth conditions: modalities) • *$M, w \vDash \Box\phi$ iff there* exits *a squeezed model $\Downarrow^k M$ satisfying both of (i) $\Downarrow^k M, \Downarrow^k w \vDash \phi$ and (ii) $\Downarrow^k M, \Downarrow^k w$ are* non-contradictory-well-behaving *with respect to ϕ.*

- *$M, w \vDash \Diamond\phi$ iff for* any *squeezed model $\Downarrow^k M$, either of the followings holds: (i) $\Downarrow^k M, \Downarrow^k w \vDash \phi$ or (ii) $\Downarrow^k M, \Downarrow^k w$ are not non-contradictory-well-behaving with respect to ϕ .*

QUITE INTUITIONISTIC. Notice that double negation fails: $\nvDash \neg\neg\phi \to \phi$ and so does law of excluded middle: $\nvDash \phi \vee \neg\phi$. Explosion still holds: $\vDash \bot \to \phi$ (for any sentence ϕ), but with a condition: our model cannot be squeezed one. This seems to entail that squeezed model can mock reasoning of *paraconsistent* logic.

FOR CLASSICAL PERSONS: CONTROL AND CHARACTERIZE! This semantics *by itself* disappoints most users for its lack of capacity to model their favorite logics. We are to find out some conditions to control behaviour of this semantics –like restriction to non-squeezed models to gain explosion– to satisfy these user's preferences [11] . But to specify necessary and sufficient conditions is not that easy.

To have classical bivalence $p \vee \neg P$, it is apparently required to have $[\![P]\!] = V(P)$ and $[\![\neg P]\!]$ complementing each other [12]. But this is not enough for a world w^* *crossing the border* such that $w^* \cap [\![P]\!] \neq \emptyset$ and $w^* \cap [\![\neg P]\!] \neq P\emptyset$. We need further conditions to control such (classically speaking) ill-behaving worlds.

Fact 1 (Condition for classicality: separation) *If our model $M = \langle L, W, V\rangle$ satisfies the following two conditions, then $M \vDash \phi$ for any classical tautologies: (1) $[\![\phi]\!]$ and $[\![\neg\phi]\!]$* separate [13] *L and (2) any $w \in W$ is* connected [14].

Corollary 1 *If the locus is* discrete [15] *and any world is connected* [16] *, any classical tautology holds.*

[10] Inspired by Heyting algebra, defining the truth condition for $\phi \to \psi$ as topological interior of disjunction: $int([\![\phi]\!]^c \cup [\![\psi]\!])$. The sense of "$\phi$ *safely* holds" is made via dimensional operation through $\Box$.

[11] Believing a certain sort of ontological commitments, this promises a certain metaphysical merit: to know what a person metaphysically commits to behind accepting some logic.

[12] There should exist $\Downarrow M$ with $\Downarrow [\![P]\!]$ and $\Downarrow [\![\neg P]\!]$ complement each other.

[13] Topological concepts *separation* and *connectedness* are defined as follows (cf. Munkres 2000). Given a topological space $\langle X, \tau\rangle$, a *separation* of X is a pair of disjoint, non-empty open subsets Y, Z whose union is X.

[14] If a space $\langle X, \tau\rangle$ does not have any separation, X is called *connected*.

[15] Some philosophers may want to claim that this discrete locus reflects the picture of how classical logicians understand the universe.

[16] Topological semantics, which takes a world to be a mere point, is an obvious consequence of this condition. In discrete space, to be connected is to be singleton.

What's the next? *Formally,* soundness and completeness (with a certain topological conditions over models) are our goal. This project still requires some inspiration or arts of finding well-working conditions until we come up with a grand scheme or an algorithm (like Sahlqvist's, converting modal sentences into the language of first order predicate classical logic; see Blackburn, Rijke, and Venema 2002, Ch. 5). Relating to proving completeness, another important task is to find some way bridging spatialist structure and some structure already known to be complete (e.g. topological model for $S4$). *Bisimulation* in spatial semantics is also desired.

The extension to *predicate* (modal) logic may provide a more interesting story for metaphysical issues [17] but surely calls intense confusion. The current version remains very mysterious metaphysically or as an applied semantic. What is the spatial entity $[\![\phi]\!]$ (and formal truth conditions/values) and how it spatially interplays with a world and its residents. Another spatial variant is worth considering. For instance, take $M, w \vDash \phi$ when $[\![\phi]\!] \subseteq w$ (cf. truth-making!).

[17] Hope spatial semantics provides a sufficient formal framework to discuss metaphysical issues on *de re* modality, e.g. trans-world identity etc.

References

Blackburn, Patrick, Maarten de Rijke, and Yde Venema (2002). *Modal Logic*. Cambridge University Press.

Munkres, James R. (2000). *Topology*. 2nd ed. Prentice Hall, Inc, p. 537.